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SELF-SIMILAR SOLUTION OF THE ANTIPLANE SHEAR FRACTURE PROBLEM IN A COUPLED FORMULATION (CREEP–DAMAGE)

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A growing antiplane shear crack in a damaged medium is considered. It is assumed that the crack tip neighbors a region of completely damaged material, in which all stress tensor components and the continuity parameter vanish. The stress–strained state is analyzed and the configuration of the region of completely damaged material is determined. The crack growth rate is estimated for various values of the constants included in the constitutive relations and kinetic equation.

Recently there have been a great number of studies in which the stress–strain state in the neighborhood of the tip of both stationary and growing cracks is determined in a coupled formulation using elasticity, plasticity, and creep theories and damage mechanics [1–12]. It is of interest to study the effect of damage accumulation on the stress and strain distribution (or creep strain rates). From a practical viewpoint, it is important to determine the rate of subcritical crack growth.

Two-dimensional problems of stationary and growing semi-infinite cracks in an infinite body in a coupled formulation (elasticity–damage and creep–damage) have a number of special features. In particular, in [1–6], it is shown that the effect of damage accumulation is manifested by the absence of stresses in the neighborhood of the crack tip or by considerable weakening of the singularity of this field. Astaf'ev et al. [1, 2] established that as the crack tip is approached, the effective stresses σ_{ij}/ψ (σ_{ij} are the stress tensor components and ψ is the Kachanov–Rabotnov continuity parameter) are limited and the continuity parameter, and the stress-tensor components decrease linearly to zero. Investigation of the resultant system of ordinary differential equations for various values of the constants m and n, included in the kinetic equation shows that the coupling of the formulation of the problem weakens the singularity (compared to the classical asymptotic relations in linear fracture mechanics) of the stress fields for small values of m and n, whereas with increase in the values of these parameters, the singularity disappears [3–6].

Another distinguishing feature of this type of problems is the existence of a region of completely damaged material, in which all stress-tensor components and continuity vanish [1, 2]. The coefficients of asymptotic expansions of the stress-tensor components (their angular distributions) and continuity were determined numerically in [1, 2]. It was established that beginning with a certain value of the polar angle φ_d (the value of $\varphi = \pi$ corresponds to the upper side of the crack and the value of $\varphi = 0$ to the crack prolongation), the function defining the principal term of the asymptotic expansion of the continuity parameter takes negative values, which contradicts the physical meaning of this quantity. Therefore, the formulation was modified. According to this formulation, a solution is sought for $0 \leqslant \varphi \leqslant \varphi_d$. The region $\varphi_d \leqslant \varphi \leqslant \pi$ localized in the neighborhood of the tip of a propagating crack is a completely damaged zone, in which all stress-tensor components and continuity are zero. On the boundary between the regions, the continuity function and the necessary stress-tensor components are continuous. Because the boundary conditions on the crack sides cannot be satisfied, Zhao and Zhang [3, 4] modify the formulation of the problem by introducing a region of completely damaged material adjoining the crack sides. In [3, 4], it is assumed that the zone of damage accumulation is adjacent to the crack tip and at infinity there is a zone in which the

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material is linearly elastic. Furthermore, in a certain region between these zones, the stress behavior is given by the asymptotic formula $\sigma_{ij} \sim r^{-1/p}$, where $p > 2$, because the stress singularity should be weaker than the singularity of the elastic solution. This hypothesis allows one to formulate the boundary condition at an infinite point, which closes the formulation of the problem, and to determine the rate of fatigue growth of the crack.

To solve problems of antiplane shear cracks, Jin and Batra [7] and Wang and Kishimoto [8] used the hodograph method assuming that the damage parameter is a function of only stresses. It is assumed that there are three regions: a region in which damage accumulation has not yet begun, a region of active damage accumulation, and a region in which damage accumulation has already completed and the continuity (or damage) parameter has reached a critical value. In [7, 8], the last region is called a saturation zone. Generally, for a propagating crack, damage also depends on space coordinates. Therefore, the kinetic equation does not admit simple integration, and, hence the hodograph method cannot be used. Murakami et al. [9] seek stress fields and the damage parameter in the neighborhood of the tip of an opening mode crack in the case of steady crack growth. However, to construct a solution, it is necessary to know the dimensions of the region of completely damaged material and its configuration. Using experimental data, Murakami et al. [9] consider the damaged region as an half-ellipse and redefine it by rays parallel to the crack sides. Thus, the geometry of the region is not determined but is specified beforehand.

In [3, 4], where the fatigue growth of an opening mode crack is studied in a coupled formulation, the boundary of the damaged zone is determined using the fact that the kinetic equation has two "branches," which define two states of the material: damage accumulation in the neighborhood of the crack tip and the absence of damage accumulation.

Stepanova and Fedina [10] and Astaf'ev et al. [11] assumed the existence of a region of completely damaged material, in which all stress-tensor components and the continuity parameter are equal to zero. Because of this zone is present near the crack tip, asymptotic expansions of the stress-tensor component and continuity parameter cannot be sought in a small neighborhood of the crack tip. Therefore, all asymptotic solutions were determined in a coordinate system shifted to the right of the tip at a distance equal to the characteristic linear dimension of the region of completely damaged material. It turned out that in the eigenfunction expansions of the continuity parameter and the stress-tensor component, the exponential terms are not related to one another. Therefore, one of the terms was specified *a priori*, which limits the generality of the problem. In [10, 11], the geometry of the region of completely damaged material is determined for various values of material constants.

In the present paper, fields of stresses, creep strain rates, and continuity are studied in a coupled formulation (creep–damage) using the self-similar variable proposed in [12] and by expanding the desired values in eigenfunctions for large distances from the crack tip. Crack growth is modelled, and the geometry of the region of completely damaged material in the neighborhood of the crack is determined.

1. Formulation of the Problem of Crack Growth in a Damaged Medium. A growing semi-infinite crack in an infinite body of a material with the constitutive relations of the coupled problem of creep theory and damage mechanics constructed with the use of the following power-law relation between creep strain rates and stresses:

$$\dot{\varepsilon}_{ij} = \frac{3}{2} B\Big(\frac{\sigma_e}{\psi}\Big)^{n-1} \frac{s_{ij}}{\psi}. \tag{1.1}$$

Here $\dot{\varepsilon}_{ij}$ are the components of the creep strain rate tensor, B and n are material constants, $\sigma_e = \sqrt{3 s_{ij} s_{ij}/2}$ is the stress intensity, ψ is the continuity parameter, and s_{ij} are the stress deviator components.

The kinetic equation defines the power law of damage accumulation:

$$\frac{d\psi}{dt} = -A\Big(\frac{\sigma_e}{\psi}\Big)^{m} \tag{1.2}$$

(t is time and A and m are material constants).

As $r \to \infty$, the asymptotic condition becomes

$$\sigma_{ij}(r \to \infty, \varphi, t) \to \tilde{C} r^{\beta} \bar{\sigma}_{ij}(\varphi, n), \tag{1.3}$$

where the exponent β is determined in the course of solution of the problem, $\bar{\sigma}_{ij}(\varphi, n)$ are functions to be determined, r and φ are polar coordinates. The case of $\beta = -1/2$ and $\tilde{C} = K_\alpha$ [$K_\alpha = K_{\rm I}$, $K_{\rm II}$, and $K_{\rm III}$ are the stress intensity factors and $\bar{\sigma}_{ij}(\varphi, n)$ are the angular distributions of the stress-tensor components of the linearly elastic problem] corresponds to the assumption that the configuration of the region of completely damaged material is defined by the singular elastic solution. The equalities $\beta = -1/(n+1)$ and $\tilde{C} = (C^*/(BI_n))^{1/(n+1)}$ (C^* is the invariant integral of the theory of steady creep and I_n is a function dependent on n and determined as a dimensionless C^*-integral) follow from the hypothesis that the geometry of the damaged zone is defined by the Hutchinson–Rice–Rosengren solution [13]. In this case, $\bar{\sigma}_{ij}(\varphi, n)$ are functions known from the Hutchinson–Rice–Rosengren solution. If $\beta = -1/(n-1)$ and $\tilde{C} = (\dot{a}/(BG))^{1/(n-1)}$ ($\dot{a}$ is the crack growth rate and G is the shear modulus), it is necessary to use the Hui–Riedel solution [14] taking into account elastic strain rates. Thus, the exponent β is considered unknown *a priori* because it is difficult to determine what asymptotic form describes the configuration of the region of completely damaged material.

By virtue of (1.3), the initial conditions are

$$\sigma_{ij}(r, \varphi, t = 0) = \tilde{C} r^{\beta} \bar{\sigma}_{ij}(\varphi, n). \tag{1.4}$$

2. Self-Similarity as a Property of the Solution of the Problem. For the constitutive relation (1.1) with the initial and boundary conditions (1.4) and (1.3) there is a self-similar variable

$$R = r\tilde{C}^{1/\beta}(At)^{1/(\beta m)}, \tag{2.1}$$

where $\beta = -1/(n+1)$ corresponds to the self-similar variable proposed in [12]. The existence of the self-similar variable R in the form of (1.2) is easily substantiated by dimensional analysis.

Indeed, the radius, time, and stress-tensor components can be reduced to dimensionless form

$$\hat{r} = rL^{-1}, \qquad \hat{t} = t\,T^{-1}, \qquad \hat{\sigma}_{ij} = \sigma_{ij}(\tilde{C}L^{\beta})^{-1},$$

where L is a certain characteristic linear dimension and T is a certain characteristic time. The relation between the characteristic length L and the time T is established by analysis of the kinetic equation (1.2): $T = \tilde{C}^{-m}L^{-\beta m}A^{-1}$. The dimensionless stresses $\hat{\sigma}_{ij}$ as functions of the dimensionless variables are written as $\hat{\sigma}_{ij}(\hat{r}, \varphi, \hat{t}) = \tilde{C}^{-1}L^{-\beta}\sigma_{ij}(rL^{-1}, \varphi, tA\tilde{C}^{m}L^{\beta m})$. This implies the existence of the self-similar variable (2.1). In this case, the stress-tensor components and the continuity parameter are written as

$$\sigma_{ij}(r, \varphi, t) = (At)^{-1/m}\hat{\sigma}_{ij}(R, \varphi), \qquad \psi(r, \varphi, t) = \hat{\psi}(R, \varphi),$$

where $\hat{\sigma}_{ij}(R, \varphi)$ and $\hat{\psi}(R, \varphi)$ are dimensionless functions of the dimensionless variable R and φ determined by solving particular boundary-value problems.

3. Antiplane Shear of Space with a Semi-Infinite Crack (Self-Similar Solution of the Coupled Problem). We consider the problem of a semi-infinite antiplane shear crack under conditions of creep in a damaged medium. From the results of [1, 2], it is assumed that near the crack tip there is a region of completely damaged material, in which all stress-tensor components and continuity vanish.

Thus, it is necessary to find a solution of the system of equations consisting of the equilibrium equation

$$\frac{\partial}{\partial R}(R\hat{\tau}_{rz}) + \frac{\partial \hat{\tau}_{\varphi z}}{\partial \varphi} = 0 \tag{3.1}$$

and the compatibility relations formulated for creep strain rates $\hat{\gamma}_{\varphi z}$ and $\hat{\gamma}_{rz}$:

$$\frac{\partial}{\partial R}(R\hat{\gamma}_{\varphi z}) = \frac{\partial \hat{\gamma}_{rz}}{\partial \varphi}. \tag{3.2}$$

Here $\hat{\gamma}_{sz} = (\hat{\tau}/\hat{\psi})^{n-1}\hat{\tau}_{sz}/\hat{\psi}$, $\hat{\tau} = \sqrt{(\hat{\tau}_{rz})^2 + (\hat{\tau}_{\varphi z})^2}$, $\hat{\gamma}_{sz}(R, \varphi) = 2\gamma_{sz}(r, \varphi, t)(At)^{n/m}/(3B)$ $(s = r, \varphi)$, and the kinetic equation

$$R\,\frac{\partial \hat{\psi}}{\partial R} = -\beta m\Big(\frac{\hat{\tau}}{\hat{\psi}}\Big)^{m}. \tag{3.3}$$

The solution of system (3.1)–(3.3) should satisfy the following boundary conditions: the condition of absence of surface forces on the crack sides

$$\hat{\tau}_{\varphi z}(R, \varphi = \pi) = 0 \tag{3.4}$$

and the symmetry condition on the crack prolongation

$$\hat{\tau}_{rz}(R, \varphi = 0) = 0. \tag{3.5}$$

As $R \to \infty$, the asymptotic condition becomes

$$\hat{\tau}_{sz}(R \to \infty, \varphi) \to R^{\beta}\bar{\tau}_{sz}(\varphi, n) \tag{3.6}$$

(boundary condition at an infinite point).

A solution of system (3.1)–(3.3) subject to boundary conditions (3.4)–(3.6) is sought over the entire plane except in the completely damaged zone, inside which the material behavior is not described by the equations formulated. It is assumed that inside the region of completely damaged material, all stress-tensor components and continuity vanish, and on the boundary of the completely damaged zone, the sought-for solution should obey the continuity conditions $\hat{\psi} = 0$ and $\hat{\tau}_{sz} = 0$ (here and below hat is omitted).

Below we describe the method of eigenfunction expansion for arbitrary n and $m = 0.7n$. A solution of system (3.1)–(3.3) subject to boundary conditions (3.4)–(3.6) is sought in the form of power expansions with the principal terms

$$(\tau_{sz}/\psi)(R,\varphi) = R^{\beta} f_{sz}^{(0)}(\varphi) + \ldots, \qquad \psi(R,\varphi) = 1 - \ldots \quad (\beta < 0) \tag{3.7}$$

in the limit $R \to \infty$, moving from the infinite point to the crack tip.

The functions $f_{sz}^{(0)}(\varphi)$ are obtained from the solution of system (3.1)–(3.3) subject to boundary conditions (3.4)–(3.6). Substituting the principal terms of the asymptotic expansions (3.7) into the equilibrium equation and consistency relation, we obtain the following system of two ordinary differential equations:

$$(f_{\varphi z}^{(0)})' = -(\beta+1) f_{rz}^{(0)}, \qquad (f_{rz}^{(0)})' = f_{\varphi z}^{(0)} \frac{(\beta n + 1) f^2 + (n-1)(\beta+1)(f_{rz}^{(0)})^2}{f^2 + (n-1)(f_{rz}^{(0)})^2}, \tag{3.8}$$

where $f(\varphi) = \sqrt{(f_{rz}^{(0)})^2 + (f_{\varphi z}^{(0)})^2}$ subject to the boundary conditions $f_{rz}^{(0)}(0) = 0$ and $f_{\varphi z}^{(0)}(\pi) = 0$.

In constructing a numerical solution of system (3.8), the boundary condition at $\varphi = \pi$ is replaced by the initial condition $f_{\varphi z}^{(0)}(0) = c$ with $\varphi = 0$. By virtue of homogeneity of system (3.8), it is possible to adopt the normalization condition $f_{\varphi z}^{(0)}(0) = 1$. Thus, the initial conditions become

$$f_{rz}^{(0)}(0) = 0, \qquad f_{\varphi z}^{(0)}(0) = 1. \tag{3.9}$$

Solving system (3.8) subject to boundary conditions (3.9) as an eigenvalue problem using the Runge–Kutta–Feldberg method, it is possible to find a numerical solution (for example, the value of $\beta = -1/(n+1)$ corresponds to the well-known Hutchinson–Rice–Rosenberg solution [13]).

The kinetic equation (3.3) leads to a binomial asymptotic expansion of the continuity parameter $\psi(R,\varphi) = 1 - R^{\beta m} f^m(\varphi)$.

Because on the boundary of the completely damaged region, the continuity parameter vanishes: $\psi = 1 - R^{\beta m} f^m = 0$, the geometry of this region can be estimated as: $R(\varphi) = [f(\varphi)]^{-1/\beta}$.

Considering the terms of the asymptotic expansion next to the principal term, it is possible to refine the geometry of the completely damaged region.

The binomial asymptotic expansion of the effective stress tensor components is sought in the form

$$(\tau_{sz}/\psi)(R,\varphi) = R^{\beta} f_{sz}^{(0)}(\varphi) + R^{\beta_1} f_{sz}^{(1)}(\varphi) + \ldots \ . \tag{3.10}$$

In what follows, we shall use the asymptotic expansions to the stress tensor components and creep strain rates:

$$\tau_{sz}(R,\varphi) = R^{\beta} f_{sz}^{(0)} - R^{\beta(1+m)} f_{sz}^{(0)} f^m + R^{\beta_1} f_{sz}^{(1)},$$
$$\gamma_{sz}(R,\varphi) = f^{n-1}\{R^{\beta n} f_{sz}^{(0)} + R^{\beta(n-1)+\beta_1}[f_{sz}^{(1)} + (n-1) f_{sz}^{(0)} f_1 f^{-2}]\}.$$

The stress-tensor components contain terms with the exponents β, β_1, and $\beta(1+m)$ of R. Three cases are possible: $\beta_1 < \beta(1+m)$, $\beta_1 > \beta(1+m)$, and $\beta_1 = \beta(1+m)$. In the first case, the new functions of the asymptotic expansion of the stress-tensor component are ignored, and in the second case, the term containing the function of angular distributions of the stress-tensor components included in the principal term of the asymptotic expansion is ignored.

We take into account all terms and assume that the terms containing R^{β_1} and $R^{\beta(1+m)}$ have the same order of magnitude, and, hence, $\beta_1 = \beta(1+m)$.

Substituting the asymptotic expansion (3.10) into the equilibrium equation (3.1) and the compatibility relation (3.2), we obtain a system of four ordinary differential equations, namely, system (3.8) and two new differential equations for the two functions $f_{sz}^{(1)}$:

$$(f_{\varphi z}^{(1)})' = m f_{\varphi z}^{(0)} f^{m-2}[f_{rz}^{(0)}(f_{rz}^{(0)})' + f_{\varphi z}^{(0)}(f_{\varphi z}^{(0)})'] + (f_{\varphi z}^{(0)})' f^m - (\beta_1 + 1)[f_{rz}^{(1)} - f^m f_{rz}^{(0)}],$$

$$(f_{rz}^{(1)})' = \{(1+\beta n+\beta m)[f^2 f_{\varphi z}^{(1)} + (n-1) f_1 f_{\varphi z}^{(0)}] - (n-1) f_1 (f_{rz}^{(0)})'$$

$$- (n-1)[f_{rz}^{(0)}(f_{rz}^{(0)})' + f_{\varphi z}^{(0)}(f_{\varphi z}^{(0)})'][f_{rz}^{(1)} + (n-1) f^{-2} f_1 f_{rz}^{(0)}] \qquad (3.11)$$

$$-(n-1)[f_{rz}^{(1)}(f_{rz}^{(0)})' + f_{\varphi z}^{(1)}(f_{\varphi z}^{(0)})' + f_{\varphi z}^{(0)}(f_{\varphi z}^{(1)})'] f_{rz}^{(0)} - 2(n-1) f^{-2} f_1 f_{rz}^{(0)} [f_{rz}^{(0)}(f_{rz}^{(0)})' + f_{\varphi z}^{(0)}(f_{\varphi z}^{(0)})']\}[f^2 + (n-1)(f_{rz}^{(0)})^2]^{-1}.$$

Here $f_1(\varphi) = f_{rz}^{(0)} f_{rz}^{(1)} + f_{\varphi z}^{(0)} f_{\varphi z}^{(1)}$. The boundary conditions are written as

$$f_{rz}^{(0)}(0) = 0, \qquad f_{\varphi z}^{(0)}(\pi) = 0, \qquad f_{rz}^{(1)}(0) = 0, \qquad f_{\varphi z}^{(1)}(\pi) = 0.$$

Solving system (3.8), (3.11) by the Runge–Kutta–Feldberg method and using the shooting method [selecting the condition $f_{\varphi z}^{(1)}(0) = c$ such that for $\varphi = \pi$ the boundary condition $f_{\varphi z}^{(1)} = 0$] is satisfied, we obtain a solution that satisfies the initial conditions

$$f_{rz}^{(0)}(0) = 0, \qquad f_{\varphi z}^{(0)}(0) = 1, \qquad f_{rz}^{(1)}(0) = 0, \qquad f_{\varphi z}^{(1)}(0) = c.$$

From the kinetic equation (3.3) and the asymptotic expansion (3.10)

$$R\,\frac{\partial \psi}{\partial R} = -\beta m R^{\beta m} f^m (1 + m R^{\beta_1 - \beta} f^{-2} f_1),$$

we obtain the trinomial asymptotic expansion of the continuity parameter $\psi(R,\varphi) = 1 - R^{\beta m} f^m - m R^{2\beta m} f^{m-2} f_1/2$.

The geometry of the completely damaged region is defined by the equation

$$R(\varphi) = [(f^m + \sqrt{f^{2m} + 2m f^{m-2} f_1})/2]^{-1/(\beta m)}.$$

To refine the configuration of the damaged region, it is necessary to construct the following terms in the asymptotic expansion of the effective stress tensor components:

$$(\tau_{sz}/\psi)(R,\varphi) = R^{\beta} f_{sz}^{(0)}(\varphi) + R^{\beta_1} f_{sz}^{(1)}(\varphi) + R^{\beta_2} f_{sz}^{(2)}(\varphi) + \ldots\ . \qquad (3.12)$$

The asymptotic expansions for the stress-tensor components and strain rates are written as

$$\tau_{sz} = R^{\beta} f_{sz}^{(0)} + R^{\beta_1}(f_{sz}^{(1)} - f_{sz}^{(0)} f^m) - R^{\beta(1+2m)}(f_{sz}^{(1)} f^m + m f_{sz}^{(0)} f^{m-2} f_1/2) + R^{\beta_2} f_{sz}^{(2)},$$

$$\gamma_{sz} = f^{n-1}\{R^{\beta n} f_{sz}^{(0)} + R^{\beta(n-1)+\beta_1}[f_{sz}^{(1)} + (n-1) f_{sz}^{(0)} f_1 f^{-2}]$$

$$+ R^{\beta(n-1)+\beta_2}[f_{sz}^{(2)} + f_{sz}^{(0)} f_2 f^{-2}] + (n-1) R^{2\beta_1 - \beta} f_{sz}^{(1)} f_1 f^{-2}\}.$$

By analogy with the binomial asymptotic expansion of the stress-tensor components, it is assumed that the terms containing R^{β_2} and $R^{\beta(1+2m)}$ have the same order of smallness and, hence, $\beta_2 = \beta(1+2m)$.

Substituting (3.12) into the equilibrium equation (3.1) and the compatibility relation (3.2), we obtain a system of six ordinary differential equations, namely, system (3.8), (3.11) and two new differential equations for the two functions $f_{sz}^{(2)}$:

$$(f_{\varphi z}^{(2)})' = -(\beta_2+1)[f_{rz}^{(2)} - m f_{rz}^{(0)} f^{m-2} f_1/2 - f_{rz}^{(1)} f^m] + (f_{\varphi z}^{(1)})' f^m + m (f_{\varphi z}^{(0)})' f^{m-2} f_1/2$$

$$+ m(m-2) f_{\varphi z}^{(0)} f^{m-4} f_1 [f_{rz}^{(0)}(f_{rz}^{(0)})' + f_{\varphi z}^{(0)}(f_{\varphi z}^{(0)})']/2$$

$$+ m f_{\varphi z}^{(0)} f^{m-2}[(f_{rz}^{(0)})' f_{rz}^{(1)} + (f_{rz}^{(1)})' f_{rz}^{(0)} + (f_{\varphi z}^{(0)})' f_{\varphi z}^{(1)} + (f_{\varphi z}^{(1)})' f_{\varphi z}^{(0)}]/2 + m f_{\varphi z}^{(1)} f^{m-2}[f_{rz}^{(0)}(f_{rz}^{(0)})' + f_{\varphi z}^{(0)}(f_{\varphi z}^{(0)})']/2,$$

$$(f_{rz}^{(2)})' = \{(1+\beta n+2\beta m)[f^2 f_{\varphi z}^{(2)} + (n-1) f_1 f_{\varphi z}^{(1)} + (n-1) f_2 f_{\varphi z}^{(0)}]$$

$$- [f_{rz}^{(1)}(f_{rz}^{(1)})' + f_{\varphi z}^{(1)}(f_{\varphi z}^{(1)})' + f_{rz}^{(2)}(f_{rz}^{(0)})' + f_{\varphi z}^{(2)}(f_{\varphi z}^{(0)})' + f_{\varphi z}^{(0)}(f_{\varphi z}^{(2)})'](n-1) f_{rz}^{(0)} \qquad (3.13)$$

$$- (n-1)[f_{rz}^{(2)} + (n-1) f^{-2} f_1 f_{rz}^{(1)} + (n-1) f^{-2} f_2 f_{rz}^{(0)}][f_{rz}^{(0)}(f_{rz}^{(0)})' + f_{\varphi z}^{(0)}(f_{\varphi z}^{(0)})']$$

$$- (n-1)[f_1 (f_{rz}^{(1)})' + f_2 (f_{rz}^{(0)})'] - (n-1) f_{rz}^{(1)}[f_{rz}^{(1)}(f_{rz}^{(0)})' + f_{rz}^{(0)}(f_{rz}^{(1)})' + f_{\varphi z}^{(1)}(f_{\varphi z}^{(0)})' + f_{\varphi z}^{(0)}(f_{\varphi z}^{(1)})']$$

$$+ 2(n-1) f^{-2}(f_1 f_{rz}^{(1)} + f_2 f_{rz}^{(0)})[f_{rz}^{(0)}(f_{rz}^{(0)})' + f_{\varphi z}^{(0)}(f_{\varphi z}^{(0)})']\}[f^2 + (n-1)(f_{rz}^{(0)})^2]^{-1}.$$

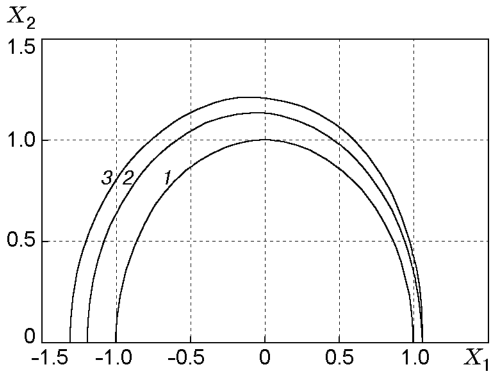


Fig. 1

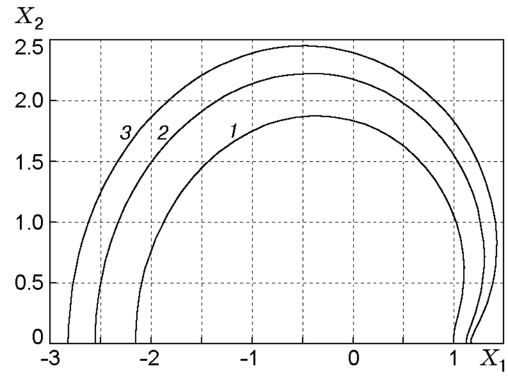


Fig. 2

Fig. 1. Geometry of the region of completely damaged material for $n = m = 1$: curve 1 is the boundary of the region defined by the binomial asymptotic expansion of the continuity parameter, curve 2 is the boundary of the region defined by the trinomial asymptotic expansion of the continuity parameter, and curve 3 is the boundary of the region defined by the four-term asymptotic expansion of the continuity parameter.

Fig. 2. Geometry of the region of completely damaged material for $n = 3$ and $m = 0.7n$ (notation same as in Fig. 1).

TABLE 1

n	m	β	n	m	β
1	1	−1.5000	6	$0.7n$	−1.1495
2	$0.7n$	−1.2303	7	$0.7n$	−1.1455
3	$0.7n$	−1.1830	8	$0.7n$	−1.1425
4	$0.7n$	−1.1648	9	$0.7n$	−1.1405
5	$0.7n$	−1.1553	10	$0.7n$	−1.1390

Here $f_2 = (f_{\varphi z}^{(1)})^2/2 + (f_{rz}^{(1)})^2/2 + f_{rz}^{(0)} f_{rz}^{(2)} + f_{\varphi z}^{(0)} f_{\varphi z}^{(2)}$. The boundary conditions are written as

$$f_{rz}^{(0)}(0) = 0, \quad f_{\varphi z}^{(0)}(\pi) = 0, \quad f_{rz}^{(1)}(0) = 0, \quad f_{\varphi z}^{(1)}(\pi) = 0, \quad f_{rz}^{(2)}(0) = 0, \quad f_{\varphi z}^{(2)}(\pi) = 0.$$

By analogy with the binomial asymptotic expansion, we obtain a solution that satisfies the boundary conditions

$$f_{rz}^{(0)}(0) = 0, \quad f_{\varphi z}^{(0)}(0) = 1, \quad f_{rz}^{(1)}(0) = 0, \quad f_{\varphi z}^{(1)}(0) = c, \quad f_{rz}^{(2)}(0) = 0, \quad f_{\varphi z}^{(2)}(0) = c_1.$$

For the problem considered, it is natural to set $\beta = -1/(n+1)$. However, it is established that for $\varphi = \pi$, the boundary conditions for $n = m = 1$ are not satisfied: irrespective of the choice of the initial value c_1, the function $f_{\varphi z}^{(2)}$ takes the same value different from zero for $\varphi = \pi$, which is confirmed by both analytical and numerical solutions for this case. Therefore, $\beta \neq -1/(n+1)$.

From the kinetic equation (3.3), we obtain

$$R \frac{\partial \psi}{\partial R} = -\beta m R^{\beta m} f^m (1 + m R^{\beta_1 - \beta} f^{-2} f_1 + m R^{\beta_2 - \beta} f^{-2} f_2),$$

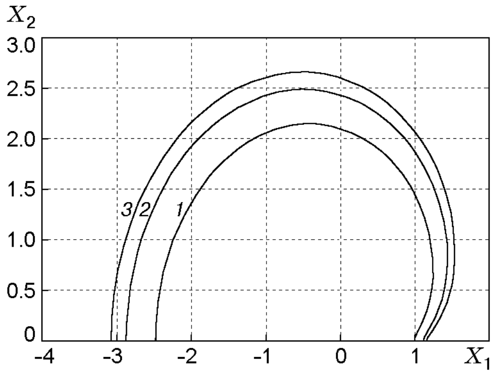
Then, the four-term asymptotic expansion of the continuity parameter becomes

$$\psi(R, \varphi) = 1 - R^{\beta m} f^m - m R^{2\beta m} f^{m-2} f_1/2 - m R^{3\beta m} f^{m-2} f_2/3.$$

The geometry of the completely damaged zone is defined by the equation

$$z^3 - z^2 f^m - m z f^{m-2} f_1/2 - m f^{m-2} f_2/3 = 0 \qquad (z = R^{-\beta m}). \tag{3.14}$$

It should be noted that for $n = m = 1$, the cubic equation (3.14) has one negative real root and two complex conjugate roots, which contradicts the physical meaning of the self-similar variable. The regions constructed on the basis of the binomial and trinomial expansions of the continuity parameter differ substantially [the characteristic linear dimension $R(0)$ differs by a factor of almost two], which is further proof of the statement $\beta \neq -1/(n+1)$.



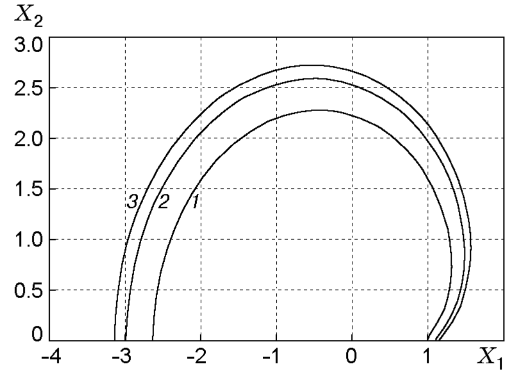


Fig. 3

Fig. 4

Fig. 3. Geometry of the region of completely damaged material for $n = 5$ and $m = 0.7n$ (notation same as in Fig. 1).

Fig. 4. Geometry of the region of completely damaged material for $n = 7$ and $m = 0.7n$ (notation same as in Fig. 1).

The eigenvalue β is determined from the condition of "convergence" of the completely damaged regions obtained on the basis of the binomial, trinomial, and four-termed expansions of the continuity parameter. Table 1 lists eigenvalues β obtained by numerical analysis of system (3.13) for various values of the material constants n and m that lead to "convergent" regions of completely damaged material. The configurations of the completely damaged regions for the values of β obtained are shown in Figs. 1–4 [$X_1 = x_1\tilde{C}^{1/\beta}(At)^{1/(\beta m)}$ and $X_2 = x_2\tilde{C}^{1/\beta}(At)^{1/(\beta m)}$, where x_1 and x_2 are Cartesian coordinates with origin at the crack tip].

4. Estimate of the Crack Growth Rate. Returning to the dimensional variables, it is possible to evaluate the dimensions of the region of completely damaged material $r = R(0)\tilde{C}^{-1/\beta}(At)^{-1/(\beta m)}$. Then, the crack growth rate becomes

$$\frac{dr}{dt} = -\frac{1}{\beta m}\, R(0)\tilde{C}^{-1/\beta}A^{-1/(\beta m)}t^{-(1+1/(\beta m))},$$

i.e., at the initial time, the crack growth rate tends to infinity, which corresponds to instantaneous start of the crack. The growth rate decreases with time, tending to zero in the limit ($t \to \infty$).

Conclusions. The asymptotic form of the far stress field, which determines the configuration of the region of completely damaged material, is established in a coupled formulation of creep theory and damage mechanics.

Asymptotic eigenfunction expansions of the effective stress tensor components and the continuity parameter are constructed, and the configuration of the damaged zone for various exponents of the degrees of the kinetic equation of damage accumulation and the power law of creep is determined in the coupled formulation (creep–damage).

The crack growth rate is estimated. It is shown that at the initial time, the crack growth rate tends to infinity, which corresponds to instantaneous start of the crack, and with time, it decreases, tending to zero in the limit.

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